Int. J. Solids Structures, Vol. 11, pp. 1347–1356. Pergamon Press 1975. Printed in Great Britain

AXISYMMETRIC BUCKLING OF CERTAIN ANNULAR COMPOSITE PLATES

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(*Received* 18 *February* 1975; *revised* 5 *May* 1975)

Abstract—Axisymmetric buckling of polar-orthotropic quasi-heterogeneous laminated annular plates are studied. A closed type solution is obtained for the case of two particular ratios of inner and outer pressures in terms of first and second kind Bessel functions of fractional order and Lommel functions. Two ratios of pressures are chosen in such a way as to represent a solution of the second order differential equation for the stress resultant function when only one of the fundamental solutions is left. The general procedure presented herein is valid for symmetrically laminated plates as well as for plates composed of materials with proportional properties. Numerous examples are shown indicating the effect of plates' heterogeneity on their elastic stability.

NOTATION

- A expression defined in eqn (12)
- A_{jk} elastic area
- A^*_{jk} extensional rigidity
- B expression defined in eqn (12)
- B_{jk} elastic static moment
- C undetermined constant
- D_{jk} elastic moment of inertia
- E_{jk} elastic stiffness moduli
- $E_\gamma, F_\gamma, G_\gamma$ linear functionals
- J_n Bessel function of first kind and order n
- k plate orthotropy parameter in bending
- $L(\,)$ differential operator defined in eqn (18)
- N_i, N_o radial compressions
- r_i, r_o inner and outer radii, respectively
- s plate orthotropy parameter in stretching
- $s_{m,n} S_{m,n}$ Lommel functions
- w transverse displacement of buckled plate
- Y_n Bessel function of second kind and order n
- β slope in radial direction of buckled plate
- η geometric parameter
- ψ stress resultant function.

1. INTRODUCTION

Several studies of buckling of *homogeneous*, circular plates under radial compression are reported in the literature. The axisymmetric buckling problem has been solved by Woinowsky-Krieger [1] in the form of Bessel functions. Later on Pandalai and Patel [2] applied a power series expansion method to the same problem. The asymmetric buckling problem was solved by Mossakowski [3] in the form of hyper-geometric functions by the method of Frobenius. Recently, Swamidas and Kunukkasseril [4] examined the axisymmetric and first asymmetric buckling of full orthotropic plates formulating the displacement of the buckled plate in terms of Bessel and Lommel functions.

The buckling of *heterogeneous* plates has also obtained considerable attention. The axisymmetric equilibrium and buckling theory for heterogeneous orthotropic circular and annular plates was established by Stavsky in [5]. A closed type solution was given by Stavsky and Friedland [6] for the elastic stability equations of isotropic composite circular plates. The buckling problem of orthotropic circular plates, which are quasi-heterogeneous in the thickness direction, was solved by Stavsky [7].

The solution of the axisymmetric buckling problem of annular orthotropic plates subjected to arbitrarily specified inner and outer pressures is yet unknown.

In the present study a closed-type solution is given for axisymmetric buckling of quasi-heterogeneous annular plates, for two specific ratios of outer/inner radial compressions. The solution of corresponding homogeneous orthotropic plates is obtained from the general solution by specialization.

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2. STATEMENT OF PROBLEM

The axisymmetric stability and thermal-buckling equations for heterogeneous orthotropic plates, formulated by Stavsky[7], are first specialized to the buckling of composite *quasi-heterogeneous* annular plates under radial compressions. Setting in eqns (2.29), (2.49) and (2.50) of reference [7],

$$p_z = 0,\ L_{23} = L_{24} = 0,\ M_{rT} = M_{\theta T} = 0, \tag{1}$$

the following two simultaneous equations, describing the neutral equilibrium of the annular plate, in terms of the slope β and the stress function Ψ are obtained:

$$\begin{aligned} A^*_{\theta\theta}(r^2\Psi'' + r\Psi') - A^*_{rr}\Psi &= 0, \\ D_{rr}(r^2\beta'' + r\beta') - D_{\theta\theta}\beta &= r\Psi\beta + rC, \end{aligned} \tag{2}$$

where a prime denotes differentiation with respect to r and the elastic coefficients are defined by the following integrals over the plate thickness

$$(A_{jk}, D_{jk}) = \int_{-h_0}^{h-h_0} (1, z^2)E_{jk}\,\mathrm{d}z, \quad (j,k = r,\theta) \tag{3}$$

where $[A^*] = [A^{-1}]$ and C is an undetermined constant. The total plate thickness is h, and h_0, given by the expression

$$h_0 = \int_0^h E_{jk}z\,\mathrm{d}z\left(\int_0^h E_{jk}\,\mathrm{d}z\right)^{-1}, \quad (j,k = r,\theta) \tag{4}$$

defines the distance from the plate's bottom face to the neutral plane. The plate under consideration is *quasi-heterogeneous*, i.e. a reference plane exists in the plate with respect to which *all* elastic statical moments vanish

$$B_{jk} \equiv \int_{-h_0}^{h-h_0} E_{jk}z\,\mathrm{d}z = 0. \tag{5}$$

The plate strain-displacement relations, the expressions for plate curvatures, axial displacement, and the resultants in terms of the stress function Ψ are given by

$$\begin{aligned} &\epsilon_{ro} = u',\ \epsilon_{\theta o} = u/r,\ \kappa_r = \beta',\ \kappa_\theta = \beta/r, \\ &w = -\int_{r_i}^{r} \beta\,\mathrm{d}r + C_4 \\ &N_r = \Psi/r,\ N_\theta = \Psi' \end{aligned} \tag{6}$$

where C_4 is an undetermined constant.

The bending moments are independent of Ψ, for the class of quasi-heterogeneous plates considered, and are related to the curvatures by

$$\begin{aligned} M_r &= D_{rr}\kappa_r + D_{r\theta}\kappa_\theta, \\ M_\theta &= D_{\theta r}\kappa_r + D_{\theta\theta}\kappa_\theta \end{aligned} \tag{7}$$

The vertical shear force is given by

$$V = C/r, \tag{8}$$

C being the same constant as in differential eqn (2.2). Thus C is the coefficient of proportionality between V and of inverse of r. From this it can be concluded that C vanishes for regularity or boundary conditions for which V vanishes. For other cases, such as those considered in this study, C has to be taken into account in the stability analysis.

Equations (2) are supplemented by three conditions at each boundary point which could be written in general form

$$E_\gamma(\Psi)=0,\ F_\gamma(w)=0,\ G_\gamma(w)=0,\ \gamma=r_i, r_0 \tag{9}$$

where E_γ, F_γ and G_γ are some linear functionals.

Let us specify the form of the functionals E_γ. The plate is compressed at the inner and outer edges by the in-plane compressions N_i, N_0, respectively. So, for E_γ we have

$$E_{r_i}(\Psi)=\Psi|_{r_i}+N_i r_i,\ E_{r_0}=\Psi|_{r_0}+N_0 r_0 \tag{10}$$

r_i and r_0 being the inner and outer radii, respectively.

Integration of eqn (2.1), taking boundary condition (10) into consideration, results in the following solution for Ψ:

$$\Psi=Ar^s+Br^{-s} \tag{11}$$

where

$$\begin{aligned} A&=(-N_0 r_i^{-s-1}+N_i r_0^{-s-1})R^{-1}\\ B&=(N_0 r_i^{s-1}-N_i r_0^{s-1})R^{-1}\\ R&=r_i^{-s-1}r_0^{s-1}\left[1-\left(\frac{r_i}{r_0}\right)^{2s}\right] \end{aligned} \tag{12}$$

and s is the *plate orthotropy parameter* in stretching

$$s=(A^*_{rr}/A^*_{\theta\theta})^{1/2}=(A_{\theta\theta}/A_{rr})^{1/2}. \tag{13}$$

The problem is formulated as follows: *Given the ratio* $N_0/N_i=\Omega$, *find the corresponding critical pair of* N_0 *and* N_i, *when the plate buckles axisymmetrically.* The exact solution of such an eigenvalue problem for any Ω is yet unknown.

In what follows, closed-type solutions are shown for a somewhat specialized class of *dependent* boundary conditions (10) resulting in an expression (11) for Ψ consisting of *either* an A term *or* B term, namely

$$N_i=N_0\eta^{s-1} \tag{14}$$

or

$$N_i=N_0\eta^{-s-1} \tag{15}$$

to give $B=0$ *or* $A=0$, respectively, in eqn (11). Let us denote the case when $B=0$ as "the first eigenvalue problem" and when $A=0$ as "the second eigenvalue problem". The non-dimensional geometric parameter η is defined as

$$\eta=r_i/r_0,\quad 0<\eta<1. \tag{16}$$

For example, for the first eigenvalue problem (14) the buckling eqn (2.2) becomes

$$L(\beta)+\lambda^2\rho^{s+1}\beta=t\rho \tag{17}$$

where $L(\ldots)$ is the differential operator in nondimensional form

$$L(\ldots)=\rho^2\frac{d^2(\ldots)}{d\rho^2}+\rho\frac{d(\ldots)}{d\rho}-k^2\rho(\ldots),$$
$$\rho\equiv\frac{r}{r_0},\ \lambda^2\equiv\frac{N_0 r_0^2}{D_{rr}} \tag{18}$$

λ is the nondimensional critical buckling load, t is the new constant, analogues to C in eqn (2.2) and k is the *plate orthotropy parameter* in bending

$$k = (D_{\theta\theta}/D_{rr})^{1/2}. \tag{19}$$

Nontrivial solutions of eigenvalue problems are represented by the set of eigenvalues λ_j ($j = 1, 2, \ldots$). The first eigenvalue λ_1 determines the lowest critical dependent buckling loads N_0 and N_i.

For the second eigenvalue problem (15) we obtain the following buckling equation

$$L(\beta) + \lambda^2 \rho^{-s-1} \beta = f\rho \tag{20}$$

where f denotes the analogue of C for this particular case.

For quasi-heterogeneous plates with $s = 1$, the first eigenvalue problem (14) reduces to the case of buckling of annular plates when the two edges are subjected to equal pressures (see, e.g. works of Schubert[8] and Yamaki[9] on the buckling of *isotropic homogeneous* plates for which $s = k = 1$). The second eigenvalue problem (15) reduces to the buckling of annular plates when the load ratio is $N_i/N_0 = \eta^{-2}$. This was considered for isotropic homogeneous plates by Buckens[10]. The same problem was studied for different boundary conditions by Mansfield[11] and Lizareff and Bareeva[12].

3. CLOSED TYPE SOLUTION FOR THE FIRST EIGENVALUE PROBLEM

Introducing a new independent variable

$$x = \frac{2\lambda}{1+s} \rho^{(1+s)/2} \tag{21}$$

the differential eqn (17) becomes

$$\frac{d^2\beta}{dx^2} + \frac{1}{x}\frac{d\beta}{dx} + \left[1 - \frac{1}{x^2}\frac{4k^2}{(1+x)^2}\right]\beta = t_1 x^{-2s/(1+s)} \tag{22}$$

where t_1 is a new constant. The solution of the ordinary differential eqn (21) is of the form

$$\beta = \beta_h + \beta_p \tag{23}$$

where β_h is the solution of the corresponding homogeneous equation and β_p is a particular solution of eqn (22).

The solution for β_h is

$$\beta_h = C_1 J_n(x) + C_2 Y_n(x) \tag{24}$$

where J_n and Y_n denote Bessel functions of order n of the first and second kind, respectively. The order n is related to the orthotropy measures s and x by the following expression:

$$n = \frac{2k}{1+s}. \tag{25}$$

The particular solution of eqn (22) is

$$\beta_p = C_3 s_{m,n}(x), \tag{26}$$

$$m = \frac{1-s}{1+s}, \tag{27}$$

and $s_{m,n}$ is a Lommel function[13]

$$s_{m,n}(x) = \sum_{p=0}^{\infty} \frac{(-1)^p x^{m+2p+1}}{[(m+1)^2 - n^2][(m+3)^2 - n^2]\cdots[(m+p+1)^2 - n^2]}. \tag{28}$$

As it is seen from (28) the function $s_{m,n}$ is undefined when $m+n$ or $m-n$ are odd negative integers. In our case the first restriction is satisfied automatically, because $s>0$ and $x>0$. The second restriction is not satisfied when $\alpha s - x = -\alpha - 1$, α being any positive integer or zero. In these cases, it follows from the general theory of Bessel functions (see [13], pp. 345–348), that Lommel's function $S_{m,n}(x)$

$$S_{m,n}(x) = s_{m,n}(x) + 2^{m-1}\Gamma\left(\frac{m-n+1}{2}\right)\Gamma\left(\frac{m+n+1}{2}\right) \times \left\{\sin\left[(m-n)\frac{\pi}{2}\right]J_n(x) - \cos\left[(m-n)\frac{\pi}{2}\right]Y_n(x)\right\} \quad (29)$$

is to be used.

It is noted that for the discrete values of s and x, mentioned above, $S_{m,n}$ may be represented by the following terminating series in descending powers of x:

$$S_{m,n}(x) = x^{m-1}\{1 - [(m-1)^2 - n^2]x^{-2} + [(m-1)^2 - n^2][(m-3)^2 - n^2]x^{-4} - \cdots\}. \quad (30)$$

Thus the plate's slope can be generally written:

$$\beta = C_1 J_n(x) + C_2 Y_n(x) + C_3 Z_{m,n}(x) \quad (31)$$

where

$$Z_{m,n}(x) = \begin{cases} S_{m,n}(x) \text{ for } \alpha s - x = -\alpha - 1,\ \alpha = 0, 1, 2, \ldots \\ s_{m,n}(x) \text{ for any other } s \text{ and } x \end{cases} \quad (32)$$

The plate's deflection is given by the following expression:

$$-w = C_1 \int J_n(x)x^m \,\mathrm{d}x + C_2 \int Y_n(x)x^m \,\mathrm{d}x + C_3 \int Z_{m,n}(x)x^m \,\mathrm{d}x + C_4. \quad (33)$$

Substitution of (33) into the boundary conditions (9.2) results in a system of homogeneous algebraic equations. The conventional requirement of nontriviality yields a characteristic equation

$$\det [a_{ij}]_{4\times 4} = 0 \quad (34)$$

where

$$a_{i1} = H_i\left(\int J_n x^m \,\mathrm{d}x\right), \qquad a_{i2} = H_i\left(\int Y_n x^m \,\mathrm{d}x\right),$$
$$a_{i3} = H_i\left(\int Z_{m,n} x^m \,\mathrm{d}x\right), \qquad a_{i4} = H_i(1) \quad (35)$$

and

$$H_1 = F_{r_i},\ H_2 = G_{r_i},\ H_3 = F_{r_0},\ H_4 = G_{r_0}$$

where F_γ and G_γ are linear functionals associated with the plate boundary conditions at the inner and outer radii. The characteristic eqn (34) is transcendental in nature and has an infinite number of roots. The lowest critical pressure corresponds to the first root of the characteristic determinant.

Consider now the specific case of a balanced composite plate with $s = k = 1$. The eqn (22) becomes

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}\beta}{\mathrm{d}x} + \left(1 - \frac{1}{x^2}\right)\beta = \frac{t_1}{x}. \quad (36)$$

The solution of the corresponding homogeneous equation is

$$\beta = C_1 J_1(x) + C_2 Y_1(x). \quad (37)$$

It can be seen that the particular solution is

$$\beta_p = t_1/x. \tag{38}$$

For the deflection we have

$$-w = C_1J_0(\lambda\rho) + C_2Y_0(\lambda\rho) + C_3 \ln\rho + C_4. \tag{39}$$

This expression coincides with Yamaki's formula (eqn 5 in [9]) for the deflection of buckled isotropic annular plates uniformly compressed for which also $s = x = 1$. Consequently Yamaki's results are included as a particular case of this study.

4. CLOSED TYPE SOLUTION FOR THE SECOND EIGENVALUE PROBLEM

In analogy with the previous section, we obtain the following differential equation for the *second eigenvalue problem*, except for the case that $s = 1$,

$$\nabla_n\beta = t_2x^{q-1},\ q = \frac{1+s}{1-s},\quad s \neq 1 \tag{40}$$

where t_2 is a constant, $\nabla_n(\ldots)$ denotes Bessel operator

$$\nabla_n(\ldots) = \frac{d^2(\ldots)}{dx^2} + \frac{1}{x}\frac{d(\ldots)}{dx} + \left(1 - \frac{n^2}{x^2}\right)(\ldots) \tag{41}$$

and x is a new variable in this particular case

$$x = \frac{2\lambda}{|1-s|}\rho^{\frac{1-s}{2}}. \tag{42}$$

It would be shown easily that the solution of the second eigenvalue problem can be obtained from the corresponding solution of the first eigenvalue problem only by interchanging "s" by "$-s$" and "k" by "$-k$", taking into account the eqn (42). As a result we again obtain expression (33) for the deflection of the buckled plate, but now the indices m and n are replaced by p and q, respectively, defined by

$$p = \frac{1+s}{1-s},\ q = -\frac{2k}{1-s}. \tag{43}$$

The function $Z_{p,q}(x)$ is given by

$$Z_{p,q}(x) = \begin{cases} S_{p,q}(x) & \text{for } k = 1+\alpha-\alpha s \text{ or } k = \alpha s - \alpha - 1, \\ s_{p,q}(x) & \text{for any other } s \text{ and } k. \end{cases} \tag{44}$$

The characteristic determinant is given by eqn (34) with p and q, replacing m,n, respectively, in expressions (35).

Consider the special case for which $s = 1$ and k is *not* necessarily unity. The differential eqn (20) reduced to a nonhomogeneous Euler equation

$$\rho^2\frac{d^2\beta}{d\rho^2} + \rho\frac{d\beta}{d\rho} + (\lambda^2 - k^2)\beta = f\rho$$

with the following general solution:

$$\beta = \begin{cases} C_1\rho^{\sqrt{k^2-\lambda^2}} + C_2\rho^{-\sqrt{k^2-\lambda^2}} & \text{for } k > \lambda \\ (C_1 + C_2\ln\rho)\rho^k & \text{for } k = \lambda \\ C_1\cos(\sqrt{\lambda^2-k^2}\ln\rho) + C_2\sin(\sqrt{\lambda^2-k^2}\ln\rho) & \text{for } k < \lambda \end{cases}$$

and the particular solution obtainable using the method of variation of parameters. If in addition to $s = 1$ k is also unity, we obtain the expression for the deflection of the buckled plated used by Buckens[10], Manfield[11] and Lizareff and Bareeva[12].

5. PLATES COMPOSED OF MATERIALS WITH PROPORTIONAL PROPERTIES

A simplification of the solution of Sections 3 and 4 is possible for a plate composed of materials with proportional properties. Such a plate may be treated as a heterogeneous plate composed of materials having elastic stiffness matrices as follows:

$$\begin{bmatrix} E^j_{rr} & E^j_{r\theta} \\ E^j_{\theta r} & E^j_{\theta\theta} \end{bmatrix} = p_j \begin{bmatrix} E^0_{rr} & E^0_{r\theta} \\ E^0_{\theta r} & E^0_{\theta\theta} \end{bmatrix} \tag{46}$$

where p_j are some positive constants; j denotes the layer number; E^0_{rr}, $E^0_{r\theta}$, $E^0_{\theta r}$, $E^0_{\theta\theta}$ are elastic stiffness moduli for a reference orthotropic material.

It could be shown easily that a plate having the property (46) is quasi-heterogeneous in the thickness direction and, moreover, $x = s$. Then for both the first and the second eigenvalue problems, the following equality holds

$$m + n = 1. \tag{47}$$

Using an integral representation of the Lommel function

$$s_{m,n} = \frac{\pi}{2}\left[Y_n(x) \int x^m J_n(x)\,dx - J_n(x) \int x^m Y_n(x)\,dx \right] \tag{48}$$

and, furthermore, Lommel's formula

$$J_{n+1}(x) Y_n(x) - J_n(x) Y_{n+1}(x) = 2/\pi x \tag{49}$$

in view of eqn (47), we will arrive at

$$s_{m,n} = x^{-n}. \tag{50}$$

Finally, the eqn (33) becomes

$$-w = x^m[-C_1 J_{-m}(x) - C_2 Y_{-m}(x) + (2m)^{-1} C_3 x^m] + C_4 \tag{51}$$

The elements a_{ij} of the characteristic determinant (34) change accordingly.

Note, that for the quasi-heterogeneous polar orthotropic plate with proportional properties, we have $x = s = (E^0_{\theta\theta}/E^0_{rr})^{1/2}$ which is independent of p_j and of the thickness of the various layers.

6. EXAMPLE PROBLEMS AND DISCUSSION

The numerical results were obtained on an IBM 370/165 Computer at the Technion. The plates considered were taken to be with total thickness h (= 4 cm), composed of lower and upper face layers of thicknesses h_1 and h_2, respectively, glued to an inner core layer of thickness h_2. The plates were clamped at the inner and outer radii, with functionals F_γ and G_γ as follows

$$\begin{aligned} F_{r_i}(\ldots) = 1,\ G_{r_i}(\ldots) = \left.\frac{d(\ldots)}{dr}\right|_{r=r_i}, \\ F_{r_0}(\ldots) = 1,\ G_{r_0}(\ldots) = \left.\frac{d(\ldots)}{dr}\right|_{r=r_0}. \end{aligned} \tag{52}$$

The Figs. 1 and 2 represent the buckling behavior of plates composed of materials with proportional properties.

The plates were composed of orthotropic layers having the following elastic stiffness matrix

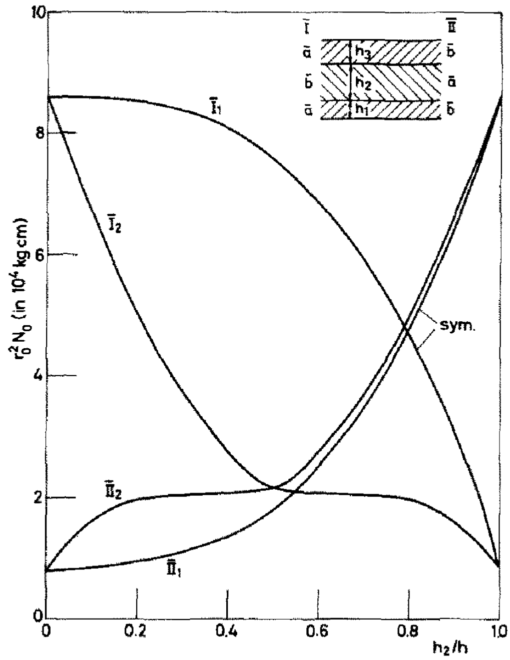

Fig. 1. Bounding buckling curves for annular plates laminated of $\bar{a}$ and $\bar{b}$ materials. The 1st eigenvalue problem ($\eta = 0{\cdot}5, p = 10$).

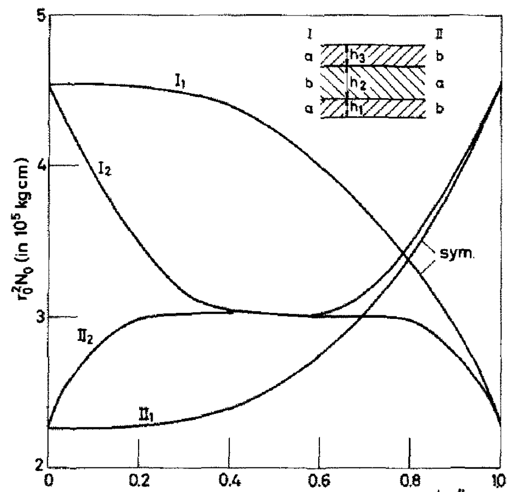

Fig. 2. Bounding buckling curves for annular plates laminates of a and b materials. The 2nd eigenvalue problem ($\eta = 0{\cdot}5, p = 2$).

as given in eqn (16), where the reference orthotropic material has the following moduli

$$E^0_{rr} = 2\times 10^6 \frac{\text{kg}}{\text{cm}^2},\ E^0_{r\theta} = E^0_{\theta r} = 0{\cdot}05\times 10 \frac{\text{kg}}{\text{cm}^2},\ E^0_{\theta\theta} = 0{\cdot}1\times 10^6 \frac{\text{kg}}{\text{cm}^2}. \tag{53}$$

For all three-layered plates considered the proportionality factors p_i in (46) are taken to be $p_1 = p_3 = 1$ and $p_2 = p$. The face layers are then made of the same material whereas the core layer is generally of a different material. Various combinations of materials "a" and "b" are considered with the following elastic moduli

$$[E]_a = \begin{bmatrix} 2 & 0{\cdot}01 \\ 0{\cdot}05 & 0{\cdot}1 \end{bmatrix} \times 10^6 \frac{\text{kg}}{\text{cm}^2},\quad [E]_b = \frac{1}{p}[E]_a. \tag{54}$$

Further, we also use "$\bar{a}$" and "$\bar{b}$" materials with properties obtained from the moduli (54) by

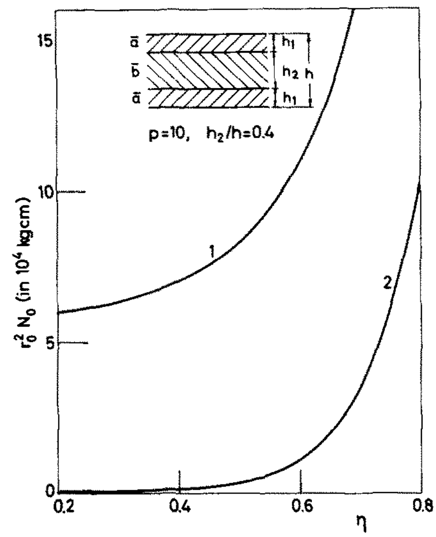

Fig. 3. Critical loads for clamped annular plates laminated of $\bar{a}$ and $\bar{b}$ materials. "1"-1st eigenvalue problem, "2"-2nd eigenvalue problem.

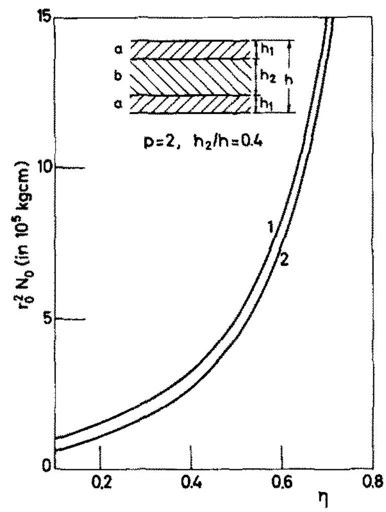

Fig. 4. Critical loads for clamped annular plates laminated of a and b materials. "1"-the 1st eigenvalue problem, "2"-the 2nd eigenvalue problem.

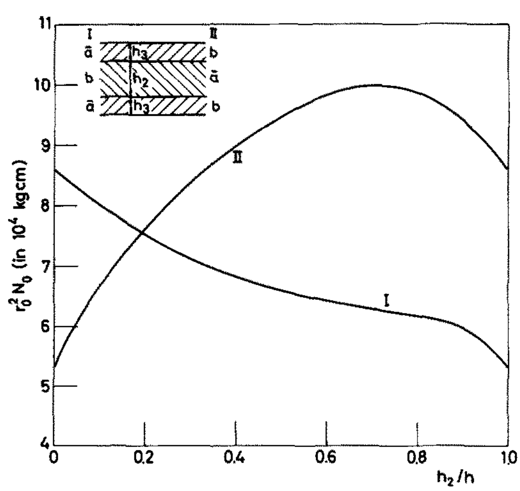

Fig. 5. Buckling curves for annular plates symmetrically laminated of $\bar{a}$ and $\bar{b}$ materials (the first eigenvalue problem $\eta = 0{\cdot}5, p = 10$).

interchanging the elastic stiffnesses in the radial and circumferential directions:

$$[E]_{\bar{a}} = \begin{bmatrix} 0\cdot1 & 0\cdot05 \\ 0\cdot05 & 2 \end{bmatrix} \times 10^6 \frac{\text{kg}}{\text{cm}^2}, \quad [E]_{\bar{b}} = \frac{1}{p}[E]_{\bar{a}}. \tag{55}$$

The radii ratio η was taken as 0·5. Fig. 1 corresponds to the first eigenvalue problem and Fig. 2 to the second one. Curves I correspond to the plates composed of $a/b/a$ layers whereas curves II refer to a reversed lay-up—$(b/a/b)$. The barred curves $\bar{\text{I}}$ and $\bar{\text{II}}$ correspond to $\bar{a}/\bar{b}/\bar{a}$ and $\bar{b}/\bar{a}/\bar{b}$, respectively. The proportionality factor p was chosen to be equal to 2 and 10 for Figs. 1 and 2, respectively. The curves I_1, II_1 are associated with a symmetric arrangement of outer layers ($h_1 = h_3$) and curves I_2 and II_2 show the results for two layered plates (either $h_1 = 0$ or $h_3 = 0$).

The curves I_2 and II_2 are the mirror reflections with respect to the $h_2/h = 0\cdot5$. As it follows from Fig. 1, the possibilities for buckling load optimization exist: for the thickness parameter $h_2/h \leq 0\cdot765$ the optimal plate is an $\bar{a}/\bar{b}/\bar{a}$ lay-up, and for $h_2/h \geq 0\cdot765$—a $\bar{b}/\bar{a}/\bar{b}$ lay-up. The ordinates of these curves correspond to maximal axisymmetric buckling loads. In Figs. 3 and 4 the critical load parameter $r_0^2 N_0$ is plotted as a function of η. An interesting implication is that the critical load tends to a limit when $\eta \to 0$. Thus, the inner pressure goes to infinity and the outer pressure tends to a constant value. This fact also is true for an isotropic one-layered plate (Ref. [13]).

In Fig. 5 the critical load parameter is given as a function of h_2/h. This figure clearly indicates that for some special lamination of composite annular plates buckling loads are greater than for individual constitutents: e.g., for the plate composed of $b/\bar{a}/b$ layers, this occurs in range $0\cdot267 \leq h_2/h \leq 1$. For $h_2/h = 0\cdot7$, use of composite plate increases the buckling load by 16·3% compared with a one-layered plate of material $\bar{a}$ and by 88·7% compared with a plate of material b.

Acknowledgements—This study was supported by the Technion Research and Development Foundation Ltd and by the Department of Mechanics. The numerical results were carried out by R. Aharoni at the Technion's Computer Center. The help of H. Arkin is also appreciated.

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